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FOUR

PROPOSITIONS, &c.

SHEWING, NOT ONLY,

That the DISTANCE of the SUN,

As attempted to be determined from

The THEORY of GRAVITY,

By a late AUTHOR,

Is, upon his OWN PRINCIPLES,

ERRONEOUS;

BUT ALSO,

That it is more than probable this

CAPITAL QUESTION

Can never be satisfactorily answered by any Calculus of the Kind.

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ADVERTISEMENT.

*TO determine the diſtance of the ſun from the earth to a tolerable degree of accuracy, has long been held a problem of a nature as difficult as it is intereſting. Hence it has from time to time employed the attention of many eminent aſtronomers; and among the reſt the preſent * Profeſſor of Mathematics in the Univerſity of Edinburgh has attempted a ſolution of it, founded upon the* Theory of Gravity. *'Tis now five years ſince his calculus was publiſhed to the world; and though his numbers differed conſiderably from thoſe which had commonly been adopted by aſtronomers, yet the author of the following propoſitions had never the ſatisfaction of ſeeing his principles publicly examined, (which the great importance of the ſubject would naturally lead one to expect) or of knowing whether they were held by the learned to be concluſive or not. Prompted however by curioſity and a natural inclination for theſe ſtudies, he now and then amuſed himſelf in the peruſal of ſuch parts of the* Doctor's Tracts *as were concerned in the enquiry; and having, as he humbly preſumed, found his calculations to be palpably wrong, and his principles*

* Dr *Stewart.*

very

very unſatisfactory, he thought it even incumbent *upon him, as a lover of truth and a well-wiſher to the ſciences, to lay his objections before the public. For ſhould the* private *papers of a few of the more eminent mathematicians have anticipated the ſubſtance of the following propoſitions, ſtill there cannot but be a claſs of ſtudents who would wiſh to ſee the enquiry reduced into one entire point of view, and to have the Doctor's principles examined in a manner pretty* eaſy *to be underſtood.—But the author hopes 'tis unneceſſary to multiply apologies for his little publication; and therefore with the following, drawn from the* time *he has choſen for its publication, he ſubmits it to the candour of his readers.—Should what he has writ prove as concluſive againſt the method of determining the diſtance of the ſun by the* Theory of Gravity, *as he perſuades himſelf it is, it muſt be a new inducement to a careful obſervation of the * approaching* Tranſit of Venus; *as from it alone this important problem ſeems capable of receiving a proper and ſatisfactory ſolution.*

Aug. 16, 1768.

* June 3, 1769.

FOUR

FOUR

PROPOSITIONS, &c.

PROPOSITION I.

SUPPOSING the orbit of the moon to be a circle, and to coincide with the plane of the ecliptic; it is required to determine the forces with which the ſun diſturbs the motion of the moon round the earth.

Let *M* repreſent the moon in its orbit *ABCD*, *E* the earth, and *S* the ſun. Draw *SM* and produce it to ☾, ſo that *SE* may be to *S*☾ as *SM*^{2} to *SE*^{2}. Then, if *SE* repreſent the gravity of the earth to the ſun, *S*☾ will repreſent the gravity of the moon to the ſun when at *M*. From the

point Q draw QN parallel to ME, meeting the line SE produced in N. Likewiſe draw ML parallel to EN, LE from the point L to the center E, and LO perpendicular to ME. The force of the moon to the ſun, repreſented by QS, may be reſolved into the two forces QN, NS, and in thoſe directions. But the line SE repreſents the force of the earth to the ſun, which if taken out of the line SN leaves NE. Hence QN and NE are the two forces which diſturb the moon in its progreſs round the earth; the joint effect of which in increaſing or decreaſing the gravity of the moon towards the earth in any part of its orbit, as M, is what we are here enquiring after.

From M draw MG perpendicular to EC, and put $SE=a$, CE or $ME=r$, EG or $MF=y$; then, from the above proportion, $SQ=\frac{SE^3\times SE}{SM^3}=$ (by ſubſtitution) $\frac{a^3}{a^3-2ay+r^3}$.

From

From SQ take $SM = \sqrt{a^2 - 2ay + r^2} = a - y$ nearly, and there remains $MQ = \frac{a^3}{a^2 - 2ay + r^2} - \overline{a - y} = \frac{3a^2y - 2ay^2 - ar^2 + r^2y}{a^2 - 2ay^2 + r^2}$. If r^2y be rejected in the numerator, and r^2 in the denominator, the value of the fraction will be very near the ſame as before, (both numerator and denominator being decreaſed at the ſame time, and the quantities rejected being much leſs than the reſt) that is, MQ will then be $= \frac{3ay - 2y^2 - r^2}{a - 2y}$ nearly; which expreſſion let us put $= z$.

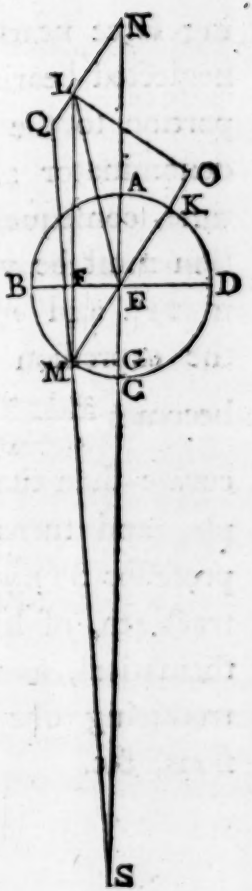
The triangles SME, MQL, are ſimilar; hence $SM = \sqrt{a^2 - 2ay + r^2} = \overline{a - y} : SE = a :: MQ = z : ML = \frac{dz}{a - y}$; and $SE = a : ME = r :: ML = \frac{az}{a - y} : QL = \frac{rz}{a - y}$. That part of the diſturbing force repreſented by QL acts in the direction QL or ME from M towards E, and conſequently increaſes the gravity of the moon towards the earth.—The force

MQ may be reſolved into the two forces QL and LM, and in thoſe directions. QL acts directly towards the earth, and therefore need not be reſolved into any other forces. The force ML and the remaining part LN are reduced into LE, and in that direction. This, again, is reſolved into LO, OE, and in theſe directions. Now the force LO is exactly balanced when the moon is at a point equidiſtant from C on the other ſide of its orbit, and therefore, taking a whole revolution together, does not diſturb its motion round the earth. The force OE therefore is what we are to eſtimate.

From the ſimilar triangles LOM, EFM, we have $MO = \frac{ML \times MF}{ME} = \frac{ayz}{r \times a-y}$, therefore $OE = \frac{ayz}{r \times a-y} - r =$ (by ſubſtituting for z its value) $\frac{3a^2y^2-2ay^3-ayr^2}{r \times a^2-3ay+2y^2} - r =$ (by reduction) to $\frac{3a^2y^2-2ay^3-a^2r^2+2ar^2y-2r^2y^2}{r \times a^2-3ay+2y^2}$.

This

This force is in the direction EM, and consequently leſſens the gravity of the moon to the earth. Hence, if the force $QL = \frac{rz}{a-y}$ = (by ſubſtitution) $\frac{3r^2ay - 2r^2y^2 - r^4}{r \times a^2 - 3ay + 2y^2}$ be taken from the above, there will then remain $\frac{3a^2y^2 - a^2r^2 - 2ay^3 - r^2ay + r^4}{r \times a^2 - 3ay + 2y^2}$; an expreſſion for the force that muſt be deducted from the gravity of the moon to the earth, in order to obtain the true force with which the moon is impell'd towards the earth when at *M*. If r^4 in the numerator, and $2y^2$ in the denominator be



ne-

neglected, and a ſtruck out of every term, we have $\frac{3ay^2-ar^2-2y^3-r^2y}{r\times a-3y}$ = the true diſturbing force nearly. (For beſides the terms neglected bearing but a very ſmall proportion to the reſt, both numerator and denominator are diminiſhed at the ſame time, conſequently the value of the fraction muſt be very little altered.) Again, if $2y^3$ and r^2y be likewiſe neglected, the expreſſion for the diſturbing force becomes $\frac{3ay^2-ar^2}{r\times a-3y}$; which, though leſs accurate than the above, is much more ſimple, and turns out the very ſame that profeſſor STEWART has found prop. 6th, tract 4th, of his Tracts Phyſical and Mathematical, as will eaſily appear, by introducing our ſymbols into his concluſions, &c.

Co-

Corollary I.

When the moon comes to K, the other end of the diameter MK, the disturbing force will become $\frac{3ay^2-ar^2+2y^3+r^2y}{r\times a+3y}$; y being here negative. Hence half the sum of these two forces will be a mean solar one, and $=\frac{3a^2y^2-a^2r^2-6y^4-3r^2y^2}{r\times a^2-9y^2}$: And if Dr Stewart's expression, found above, be reduced in like manner, we shall have $\frac{a^2}{r}\times\frac{3y^2-r^2}{a^2-9y^2}$ for the mean force; which is the same as that he has determined prop. 8th of the before-mentioned tract.

Corollary II.

The proportion between the mean solar force determined above, and the force of the moon to the earth, is easily determined. Let p represent the periodic time of the moon round the earth, P the periodic

riodic time of the earth about the ſun; then, the force of the earth to the ſun, is to the mean ſolar force, as a is to $\frac{3a^2y^2-a^2r^2-6y^4-3r^2y^2}{r\times\overline{a^2-9y^2}}$. And (by the laws of central forces) the force of the moon to the earth, is to the force of the earth to the ſun, as $\frac{r}{p^2}$ to $\frac{a}{P^2}$: Hence the force of the moon to the earth, is to the mean ſolar force, as $\frac{r}{p^2}$ is to $\frac{1}{P^2}\times\frac{3a^2y^2-a^2r^2-6y^4-3r^2y^2}{r\times\overline{a^2-9y^2}}$, or, as r to $\frac{p^2}{P^2}\times\frac{3a^2y^2-a^2r^2-6y^4-3r^2y^2}{r\times\overline{a^2-9y^2}}$.

PRO-

PROPOSITION II.

Required to determine the proportion of the variation of the ſun's diſtance from the earth, to a correſponding variation of the diſturbing force of the ſun upon the moon at any point M *in its orbit.* [*See laſt fig.*]

'Tis proved by the writers on this part of phyſical aſtronomy, that the diſtance of the moon from the earth increaſes and decreaſes, as the diſtance of the ſun decreaſes and increaſes; but an indefinitely ſmall variation of the ſun's diſtance, will ſcarce at all affect the diſtance of the moon. Therefore, in the expreſſion for the diſturbing force, viz.* $\frac{P^2}{p^2} \times \frac{3ay^2 - ar^2 - 2y^3 - r^2y}{r \times a - 3y}$, r may be looked upon as a conſtant quan-

* This will eaſily appear from cor. 2, laſt prop.

tity, and, conſequently, y will likewiſe be given for any given point M in its orbit. If therefore the expreſſion for the diſturbing force be thrown into fluxions, ſuppoſing only a variable, we ſhall have the variation of the diſturbing force = $\frac{P^2}{p^2} \times \frac{\dot{a} \times -y}{\overline{a-3y}|^2} \times \frac{7y^2-4r^2}{r}$: But the variation of the ſun's diſtance is $\dot{a}$; hence the variation of the diſturbing force is to the variation of the ſun's diſtance, as $\frac{P^2}{p^2} \times -y \times \frac{7y^2-4r^2}{r}$ to $\overline{a-3y}|^2$.

If $\frac{P^2}{p^2} \times \frac{n}{r \times a-3y}$ be neglected from the diſturbing force, (n being ſuppoſed an indefinitely ſmall negative or poſitive quantity) the variation in the ſun's diſtance, occaſioned thereby, will be found by this proportion, viz. as $\frac{P^2}{p^2} \times -y \times \overline{7y^2-4r^2}$ is to $r \times \overline{a-3y}|^2$, ſo is $\frac{P^2}{p^2} \times \frac{n}{r \times a-3y}$ to $\overline{a-3y} \times \frac{n}{-y \times \overline{7y^2-4r^2}}$ = the variation in the ſun's diſtance, conſe-

ſequent upon neglecting an indefinitely ſmall quantity (n) in the numerator of the fraction, expreſſing the diſturbing force.

Corollary I.

For any given point in the orbit, the variation in the ſun's diſtance is as $\overline{a-3y} \times n$; that is, as the ſun's diſtance multiply'd into the part neglected in the numerator, nearly. This is evident from what is gone before. If n be ſuppoſed conſtant, the variation will then be as $\overline{a-3y}$, that is, directly as the diſtance, nearly; for $3y$ is very ſmall compared with a.

Corollary II.

Let us now ſuppoſe $\frac{p^2}{p_1} \times n$ to be any quantity neglected from the expreſſion for the diſturbing force; then proportioning,

as above, there comes out $\overline{a-3y}|^2 \times \frac{rn}{\cdot y \times 7y^2 - 4r^2}$ for a correſponding variation of the ſun's diſtance; which is as the quantity neglected multiplied into the ſquare of the diſtance, nearly; but if n be conſtant, 'tis as the ſquare of the diſtance directly; for $3y$, as was obſerved before, bears only a very ſmall proportion to a.

SCHOLIUM.

From the laſt propoſition and its corollaries it will plainly appear, that the variation of the diſturbing force is extremely ſmall, when compared with a correſponding variation of the ſun's diſtance, and this ſtill leſs as the diſtance of the ſun is greater; *a concluſion directly contrary to the opinion of Dr* STEWART. For, in the preface to Tracts Phyſical and Mathematical, we have the following paragraph: "It is well known that "the

" the various methods hitherto attempted
" to ſolve this curious problem, have
" failed in a great meaſure, on account
" of the vaſt diſtance of the ſun from
" the earth; but the method hinted at
" here, will give the ſolution the more
" accurately, the more diſtant the ſun is
" from the earth; for it proceeds on the
" ſuppoſition that the diſtance of the ſun
" from the earth is great."

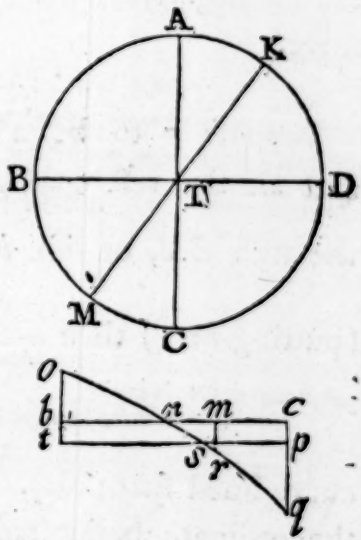
Upon a ſlight view of the matter this aſſertion looks probable, For, if any quantity (n) be neglected in the numerator of the fraction expreſſing the diſturbing force, its value in the fraction is as $\frac{n}{r \times a - 3y}$, which is evidently leſs as a, or the diſtance of the ſun from the earth, is greater: And therefore it would ſeem to follow, that it would leſs affect the diſtance of the ſun, determined by this method. But this, as we have found above, is juſt the reverſe. And here let

us conſider, whether the ſingle truth juſt demonſtrated, does not of itſelf ſufficiently prove, that the diſtance of the ſun can ſcarce ever be truly aſcertained by the *theory of gravity*.

PRO-

PROPOSITION III.

The mean ſolar force for any point of the orbit, as M, *being* $\frac{P^2}{p^2} \times \frac{3a^2y^2 - a^2r^2 - 6y^4 - 3r^2y^2}{r \times a^2 - y^2}$, (*cor.* 2 *prop.* 1); *it is required to find the ſum of all the mean forces for a quadrant of the orbit* BC.



Draw the line *bc*, equal to the arch *BC*, and let *m*, in the line, correſpond to *M*, in the arch. At *m* erect a perpendicular, and therein take *mr* equal to the mean diſturbing force at $M = \frac{P^2}{p^2} \times \frac{3a^2y^2 - a^2r^2 - 6y^4 - 3r^2y^2}{r \times a^2 - y^2}$. Let this be done

done for every point of the line *bc*, and draw the curve *qrsno* bounding theſe perpendiculars.

From this conſtruction it may eaſily be ſeen, that the curve will cut the axis *bc* in ſome point as *n*, that is, when $3a^2y^2 = a^2r^2 + 6y^4 + 3r^2y^2$; for then the ordinate $mr = 0$. From *n* to *b* the ordinates are all negative, and from *n* to *c* all poſitive.

Suppoſe *y* to be a flowing quantity, and all the reſt conſtant, the fluxion of the arch *BM*, or line $bm = \frac{r\dot{y}}{\sqrt{r^2-y^2}} = \frac{\dot{y}}{\sqrt{1-y^2}}$, (putting $r = 1$) then $\frac{\dot{y}}{\sqrt{1-y^2}} \times \frac{p^2}{p^2} \times \frac{3a^2y^2-a^2-6y^4-3y^2}{1\times a^2-9y^2}$ = fluxion of the area of the curve lined ſpace *obcq*. The expreſſion for the ordinate being reduced into a ſeries, we have $mr = \frac{p^2}{p^2} \times$

$-1+$

$$-1+\frac{3a^2-12}{a^2}y^2+\frac{21a^2-108}{a^4}y^4+\frac{189}{a^4}y^6 \text{ \&c.}$$

Therefore the fluxion of the ſpace $=\frac{\dot{y}}{\sqrt{1-y^2}}$

$$\times\frac{p^2}{p^2}\times\sqrt{-1+\frac{3a^2-12}{a^2}y^2+\frac{21a^2-108}{a^4}y^4+\frac{189}{a^4}y^6} \text{ \&c.}$$

and the fluent, when $y=1$, becomes $\frac{p^2}{p^2}\times$

$$-c+\frac{c}{2}\times\frac{3a^2-12}{a^2}+\frac{3c}{8}\times\frac{21a^2-108}{a^4}+\frac{5c}{16}\times\frac{189}{a^4} \text{ \&c.}$$

c being put = line bc or arch $BC=\frac{3.14159}{2}$. This expreſſion is equal to the difference of the two areas obn, and cnq, or $=cnq-obn$; that is, equal the ſum of the poſitive forces, that have acted upon the body in paſſing from B to C, made leſs by the negative forces.

Corollary I.

Draw the line tp parallel to bc, and at ſuch a diſtance from it, that the area $tbcp$ may be = the area $cnq-obn$; then will tb or $cp=\frac{p^2}{p^2}\times$

$-1+\frac{1}{2}\times\frac{3a^2-12}{a^2}+\frac{3}{8}\times\frac{21a^2-108}{a^4}+\frac{5}{16}\times\frac{189}{a^4}$ &c. For the baſe c, multiply'd into this perpendicular, gives the area of the parallelogram $tbcp=\frac{p^2}{p_2}\times-c+\frac{c}{2}\times\frac{3a^2-12}{a^2}$ &c. = the curve lin'd area found above. This perpendicular is therefore a *mean ſolar force*, which, when reduced, becomes $=\frac{p^2}{p_2}\times\frac{8a^4+30a^2+297}{16a^4}$, r being all along ſuppoſed equal 1.

Corollary II.

If the mean diſturbing force for the point *M*, made uſe of by *profeſſor Stewart*, be taken equal the ordinate *mr*, and the curve lined area, found and reduced in the ſame manner as above, we ſhall have the mean ſolar force $=\frac{p^2}{p_2}\times\frac{8a^4+90a^2+729}{16a^4}$.

Scho-

SCHOLIUM.

As *Dr Stewart* makes uſe of the mean ſolar force found above, for determining the ſun's diſtance, it will not be amiſs to examine what proportion a ſmall variation in the ſun's diſtance bears to a correſponding one in the mean ſolar force; and likewiſe, whether a mean force can properly be made uſe of in the enquiry at all, or not.

Let the variation of the ſun's diſtance be repreſented by $\dot{a}$, then the variation of the mean ſolar force, viz. $\frac{p^3}{P^2} \times \frac{8a^4+30a^2+297}{16a^4}$, will be found, when thrown into fluxions, $= \frac{p^3}{P^2} \times \frac{-15a^2\dot{a}-297\dot{a}}{4a^5}$. Therefore, the variation of the ſun's diſtance is to the variation of the mean ſolar force, as $4a^7 \times P^2$ to $p^3 \times \overline{-15a^2+297}$, or (by neglecting 297) as $4a^7 \times P^2$ to $-15p^3$. Hence it is

 plain

plain, that when the mean ſolar force increaſes, the diſtance of the ſun decreaſes, and *e contra*, and that in proportion as the cube of the diſtance directly.

Here again we cannot help obſerving, that the greater the value of a (or the diſtance of the ſun) is, the leſs accurate will this method of calculation be. For, ſuppoſing the variation of the mean diſturbing force to be conſtant, the variation of the ſun's diſtance will be as the cube of the diſtance, very nearly. It will likewiſe evidently appear, that, to get the ſun's diſtance to any tolerable exactneſs, the mean diſturbing force muſt be perfectly true; for the variation in this is to a correſponding variation in the ſun's diſtance, as 1 to $\frac{4a^3P^2}{15p^2}$. Taking a at the loweſt, ſuppoſe only equal 300 times the moon's diſtance, then the proportion of the variation juſt mentioned, is as 1 to $\frac{4\times 27000000}{15} \times \frac{p^2}{P^2}$,

or

or $7200000 \times 178{\cdot}725$ ($\frac{p^2}{p^2}$ being $=178{\cdot}725$.)

It will likewiſe eaſily appear, that the mean diſturbing force bears a much greater proportion to an indefinitely ſmall variation therein, than the diſtance of the ſun to a correſponding variation in the diſtance; that is, $\frac{p^2}{p^2} \times \frac{8a^4+30a^2+297}{16a^4}$ bears a greater proportion to $\frac{p^2}{p^2} \times \frac{-15a^2\dot{a}-297\dot{a}}{4a^5}$, than a to $\dot{a}$. For, dividing the antecedents by the conſequents, the meaſures of the two ratios are $\frac{a}{4a} \times \frac{8a^4+30a^2+297}{-15a^2-297}$ and $\frac{a}{\dot{a}}$; which are to each other as $8a^4+30a^2+297$ to $-4 \times 15a^2+297$; or (by neglecting 297) as $2a^2+7\frac{1}{2}$ (or $2a^2$, nearly) to -15. It will therefore by no means appear ſtrange, that the diſtance of the ſun, found by making uſe of the mean force, according to *Dr Stewart*, ſhould be much different from that which comes out from making

making uſe of the true mean force, as will appear farther on.

It will be neceſſary here to remark, that we call the mean force, determined above, the true *mean force*, only to diſtinguiſh it from *Dr Stewart*'s; for tho' it be much more accurate than his, there are too many quantities neglected to make it deſerving of its title, otherwiſe than by way of compariſon.

But to proceed with another objection. Since the perpendicular *cp* (ſee fig. p. 15) is plainly an arithmetical mean among all the perpendiculars which may be conceived erected upon the line *bc*, the mean diſturbing force, which it repreſents, muſt be an *arithmetical mean* among all the forces, which have acted in the deſcription of the quadrant. Now, were the motion of the apſes directly in the ſimple ratio of the diſturbing force, then if any number of

diſ-

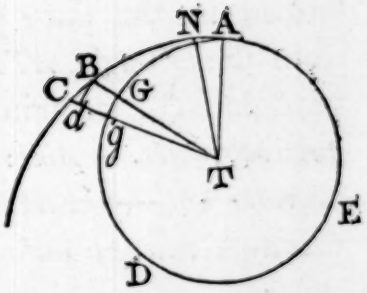
diſturbing forces, all different from one another, were to act each an equal time, 'tis evident they would occaſion the ſame motion in the apſes, as if a force, which was an arithmetical mean among theſe forces, were to act for the whole time. But it will preſently be ſeen, that the motion of the apſes is not in the direct ſimple ratio of the diſturbing force: hence it muſt be manifeſt, that a mean ſolar force, of the nature of the above, can never rightly anſwer the purpoſe for which it is intended, were it ever ſo accurately determined.

PRO-

PROPOSITION IV.

Let ADE *be a circle deſcribed round the center* T, *and ſuppoſe that a body deſcribes it by a force, tending to* T, *which is inverſely as the ſquare of the diſtance from* T; *if another force, which is as its diſtance, act upon the body directly from the center, and, by the compound force, make it deſcribe the curve* ABC; *it is required, from having the proportion of theſe two forces given, to determine the motion of the apſes, and* e contra, *from having the motion of the apſes given, to determine the proportion of the two forces.*

Put $TA = r = 1$, $TB = x$, $GB = y$, $AB = z$, $Cd = \dot{x} = \dot{y}$, $BC = \dot{z}$, and $Bd = \dot{u}$. Let the force of the body T, at the diſtance TA, be repreſented by r or 1, and the diſturbing force by n. Then, at the diſtance TB or x, the two forces that act upon the body



body moving at B will become $\frac{r^3}{x^2}$ and $\frac{nx}{r}$. The velocity of the body in the circle AGD, is the ſame as would be acquired by falling from a height above $A=AT$ or x, which therefore is eaſily found to be $=r$. [See prop. 3 tract 3, of Tracts *Phyſical and Mathematical.*] It is likewiſe commonly known, that the central force multiplied into $\dot{x}$, is equal the velocity into the fluxion of the velocity. Let the velocity at B be put $=v$, then we have this equation $\overline{\frac{r^3}{x^2}-\frac{nx}{r}}\times\dot{x}=-v\dot{v}$, and taking the fluents $-\frac{r^3}{x}-\frac{nx^2}{2r}=-\frac{v^2}{2}$. But at A the equation becomes $-\frac{r^3}{r}-\frac{nr^2}{2r}=-\frac{r^2}{2}$; therefore, by correction and reduction, we have $\frac{r^3}{x}\times\overline{2r-x}+\frac{n}{r}\times\overline{x^2-r^2}=v^2$. Suppoſe AN to be de-

 ſcribed

ſcribed in the ſame time as BC; then from the triangle ATN, being equal the triangle BTC, we gain the following proportions, viz. $AN^2 : BC^2 :: r^2 : v^2$,
and — — $Bd^2 : AN^2 :: r^2 : x^2$,
therefore, ex æquo $Bd^2 : BC^2 :: r^4 : x^2 v^2$,
that is, — — $\dot{u}^2 : \dot{z}^2 :: r^4 : x^2 v^2$.
By diviſion — $\dot{u}^2 : \dot{x}^2 :: r^4 : x^2 v^2 - r^4$,
hence — — $\dot{u} : \dot{x} :: r^2 : \sqrt{x^2 v^2 - r^4}$,
conſequently $\dot{u} = \frac{r^2 \dot{x}}{\sqrt{v^2 x^2 - r^4}}$. In the value of v^2 found above ſubſtitute $r+y$ for x, and neglecting all the terms where y is found above the ſecond power, we ſhall have, when reduced, $\dot{u} = \frac{r^2 \dot{y}}{\sqrt{2nr^2 y - r^3 - 5nrxy^2}}$
$= \frac{r^{\frac{1}{2}} y^{-\frac{1}{2}} \dot{y}}{\sqrt{2nr - r - 5nxy}}$. Therefore $\sqrt{r+y} \times \dot{u} =$
$\frac{r^{\frac{5}{2}} y^{-\frac{1}{2}} \dot{y}}{\sqrt{2nr - r - 5nxy}} + \frac{r^{\frac{3}{2}} y^{\frac{1}{2}} \dot{y}}{\sqrt{2nr - r - 5nxy}}$ = twice the area of the fluxionary triangle BTC. When the body comes to the apogee, $\dot{u} = \dot{z}$ or $\dot{y} = 0 = \dot{x}$, that is, the denominator $\sqrt{2nr - r - 5nxy} = 0$;

$=0$; therefore, ſuppoſing $r=1$, y will be found $=\frac{2n}{1-5n}$. Taking the fluent of the above fluxion at that time, and ſtill ſuppoſing $r=1$, we have twice the area deſcribed while the body paſſes from peregee to apogee, or the area deſcribed from apogee to apogee $=\frac{c}{\sqrt{1-5n}}+\frac{nc}{\overline{1-5n}|^{\frac{1}{2}}}$, ($c$ being $=3{\cdot}14159$.) Now the area of the circular orbit is $=c$; therefore the above found area is to the area of the circle, as $\frac{1}{\overline{1-5n}|^{\frac{1}{2}}}+\frac{n}{\overline{1-5n}|^{\frac{3}{2}}}$ to $1 :: 1-4n : \overline{1-5n}|^{\frac{3}{2}}$. Hence the time of moving from apogee to apogee, is to the time of deſcribing the circle ADE with the force r or $1 :: 1-4n ; \overline{1-5n}|^{\frac{3}{2}}$.

The periodic time of the body in the curve ABC, with the compound force, will be nearly the ſame as in a circle at the mean diſtance, and with the compound force proper to that diſtance. The leaſt

diſtance is r, or 1, the greateſt diſtance $= 1 + \frac{2n}{1-5n}$, half the ſum of theſe $= 1+n$, nearly, equal the mean diſtance. The compound force, at this diſtance, will be eaſily found $= \frac{1-n\times\overline{1+n}^3}{\overline{1+n}^2}$. It is a property of central forces commonly known, that the periodic times in circles, are in the ſubduplicate ratio of the radius of the circle directly, and of the central force inverſly: Therefore, the time of deſcribing the circle ADE, is to the time of deſcribing a circle at the mean diſtance, or the time of a revolution, as $\sqrt{1-n\times\overline{1+n}^3}$ to $\overline{1+n}^{\frac{3}{2}}$. Hence, compounding this proportion with that found above, we ſhall have the time from apogee to apogee, to the time of a revolution round the body T, as $\overline{1-4n}\times\sqrt{1-n\times 1+n}^3$ to $\overline{1-5n}^{\frac{3}{2}}\times\overline{1+n}^{\frac{3}{2}}$, or as $1-\frac{9}{2}n+\frac{3}{8}n^2$ to $1-6n-\frac{3}{2}n^2$, nearly. Therefore if n be given, the motion of the apſes may be found

found; and if the motion of the apſes be given, n is eaſily found: For, taking the ſame numbers as are made uſe of in prop. 9, ſect. 1. Supplement to Tracts Phyſical and Mathematical, 58091 : 57600 : : $1-\frac{9}{2}n+\frac{3}{8}n^2 : 1-6n-\frac{3}{2}n^2$, or, $\frac{58091}{57600} = 1-\frac{3}{2}n-\frac{69}{8}n^2$; hence n is eaſily found,

The ſame by a different method.

We found before that $\dot{u} = \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{2n-1-5n\times y}}$: Then by ſimilar triangles $TB : TG :: Bd : Gg$, that is, $1+y ; 1 ; : \frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{2n-1-5n\times y}} : \frac{y^{-\frac{1}{2}}}{1+y} \times \frac{\dot{y}}{\sqrt{2n-1-5n\times y}} = Gg$, which is the meaſure of the angle BTC. Throw $\frac{y^{-\frac{1}{2}}}{1+y}$ into a ſeries, then multiplying and finding the fluent, we have $\frac{c}{\overline{1-5n}^{\frac{1}{2}}} - \frac{nc}{\overline{1-5n}^{\frac{3}{2}}} + \frac{3n^2c}{2.\overline{1-5n}^{\frac{5}{2}}} - \frac{5n^3c}{2.\overline{1-5n}^{\frac{7}{2}}}$ &c. = arch deſcribed in paſſing

ſing from peregee to apogee; therefore, from what has been ſaid above, it is evident, that $58091 : 57600 :: \frac{c}{1-5n|^{\frac{1}{2}}} - \frac{nc}{1-5n|^{\frac{3}{2}}} + \frac{3n^2c}{2.\ 1-5n|^{\frac{5}{2}}}$ &c. : c, from which proportion n will eaſily be found.

Corollary.

It is a thing commonly known, and which we ſhall not ſtay here to demonſtrate, that the diſturbing force of the ſun upon the moon is, *cæteris paribus*, in the direct ſimple ratio of the moon's diſtance from the earth; therefore n may very fitly be put equal the mean diſturbing force found in Cor. 1ſt, Prop. 3d. Hence, for determining a, or the ſun's diſtance, there is this equation $\frac{p^2}{P^2} \times \frac{8a^4+30a^2+297}{16a^4} = n$, which reduced gives $a = \sqrt{\frac{15p^2}{16P^2n-8p^2} + \sqrt{\frac{15p^2}{16P^2n-8p^2}\Big|^2 + \frac{297p^2}{16P^2n-8p^2}}}$.

But

But if Dr *Stewart*'s mean force be put $= n$, we ſhall have, when reduced as above, $a = \sqrt{\frac{45p^2}{16P^2n-8p^2} + \sqrt{\frac{45p^2}{16P^2n-8p^2}}^2 + \frac{729p^2}{16P^2n-8p^2}}$. Theſe two equations will give the values of a much different, let n be what it will, as is evident from inſpection.

Scholium.

We ſhall now endeavour to point out the principal miſtakes, which, we apprehend, Dr *Stewart* has made, in attempting to ſolve this curious problem. And firſt let us examine his calculations, ſuppoſing all the principles he has gone upon to be right.

In this propoſition we had the following proportion, viz. the time of the body moving from apogee to apogee, is to the time of the body deſcribing the circle ADE with the force of the body T alone

alone :: $1-4n : \overline{1-5n}^{\frac{1}{2}}$. Then, if we ſuppoſe with Dr *Stewart*, that the time of a revolution of the body in the curve *ABC*, is equal to the time of a revolution in the circle *AGD*, with the compound force at *A*, we ſhall have the following proportion, which is the ſame with his; ſee page 381 Tracts Phyſical and Mathematical, viz. The time of deſcribing the circle, with the force of the body *T* alone, is to the time with the compound force :: $\overline{1-n}^{\frac{1}{2}} : 1$. Therefore compounding this proportion with the above, the time of moving from apogee to apogee, is to the time of revolving with the compound force :: $1-4n \times \overline{1-n}^{\frac{1}{2}} : \overline{1-5n}^{\frac{3}{2}}$; that is, 58091 : 57600 :: $\overline{1-4n \times 1-n}^{\frac{1}{2}} : \overline{1-5n}^{\frac{3}{2}}$; or (putting $a=58091$, $b=57600$) $a^2 : b^2 :: 1-9n+24n^2-16n^3 : \overline{1-5n}^3 = 1-15n+75n^2-125n^3$. From dividing the conſequents by the antecedents, we get $\frac{b^2}{a^2} = 1-6n-3n^2$, nearly; whence n will be found $= .0028015$.—Dr *Stewart*, for

for finding the value of n, has this proportion, a^2 is to b^2 as the cube of the greateſt diſtance is to the cube of the leaſt diſtance. [See page 381 Tracts, &c.] But 1 is the leaſt diſtance, and $1+\frac{2n}{1-5n}$ the greateſt diſtance; hence this proportion, $a^2 : b^2 :: \overline{1+\frac{2n}{1-5n}}|^3 : 1^3 :: \overline{1-3n}|^3 : \overline{1-5n}|^3$ $::1-9n+27n^2-27n^3 : \overline{1-5n}|^3$. Compare this with the former proportion, and it will be eaſy to ſee where they differ. For, if n^3 in each be neglected, they only vary in the third term, where there is $27n^2$ in the laſt proportion, inſtead of $24n^2$ in the former. The difference is indeed very ſmall, yet it occaſions a very great alteration in the ſun's diſtance, as will appear immediately. From this laſt proportion the value of n is $\cdot 00279770$ &c. whoſe reciprocal is $357\cdot 43365$, the very ſame number found in prop. 9 ſect. 1, ſupplement to Tracts, &c. Now if this

value of n be ſubſtituted in the ſecond equation for a, page 31, and p^2 be put = 1, and $P^2 = 178{\cdot}725$, a will come out = 495·932 &c. which falls within the limits the Doctor has preſcribed, and agrees with his number in every place, except the laſt. But if the other value of n, viz. ·0028015, be ſubſtituted in the ſame equation, a will come out = 89·8, nearly.

The difference of the two values of n, here made uſe of, is only ·0000038 &c. which is ſcarce the 737th part of the whole, and yet the difference in the ſun's diſtance occaſioned by this extremely ſmall variation in n, or the mean diſturbing force, is no leſs than 406 times the moon's diſtance; which is even more than the ſun's diſtance has generally been thought to be. But if n was taken = ·00279759, and ſubſtituted in the ſame equation, a would come out infinite, though the variation here, from Dr *Stewart's* value of n, is

n, is only ·0000011 &c. which is leſs than the 25433d part of it. This laſt value of n is $= \frac{1}{357 \cdot 45} = \frac{1}{2P^2}$, hence in the equation for the value of a, the denominator $16P^2n - 8p^2 = 0$, therefore a muſt be infinite.

To prevent any objections that may be made to the above method of calculation, we ſhall examine in what manner the value of n is affected, by rejecting all the terms in the value of v^2x^2, in which y is found above the ſecond power. [See page 26.] Our value of $\dot{u}$ is $\frac{\dot{y}}{\sqrt{2ny - 1 - 5nxy^2}}$; but if the term where y^3 is found be taken in, then $\dot{u}$ will be $= \frac{\dot{y}}{\sqrt{2ny - 1 - 5nxy^2 + 4ny^3}}$. Extracting the root, the denominator becomes $= \overline{2ny - 1 - 5nxy^2}\Big|^{\frac{1}{2}} + \frac{4ny^3}{\overline{2ny - 1 - 5nxy^2}\Big|^{\frac{1}{2}}} -$, &c. Dividing $\dot{y}$ by this

ſeries, $\dot{u}$ will $= \frac{\dot{y}}{\sqrt{2ny - \overline{1-5n} \times y^2}} - \frac{2ny^3\dot{y}}{2ny - \overline{1-5n} \times y^2|^{\frac{3}{2}}}$

&c. Multiply both ſides by $1+y$, and take the fluent when $y = \frac{2n}{1-5n}$, and we have the whole area $= \frac{c}{\overline{1-5n}|^{\frac{1}{2}}} + \frac{nc}{\overline{1-5n}|^{\frac{3}{2}}}$ — ſome ſmall quantity, which put $= s$. This area therefore is to the area of the circle, as $\frac{c}{\overline{1-5n}|^{\frac{1}{2}}} + \frac{nc}{\overline{1-5n}|^{\frac{3}{2}}} - s$ is to c, or as $1-4n-s \times \overline{1-5n}|^{\frac{3}{2}}$ is to $\overline{1-5n}|^{\frac{3}{2}}$. And the time from apogee to apogee, is to the time of a revolution in the circle *AGD* with the force of the body T alone, as $\overline{1-4n-s \times 1-5n}|^{\frac{3}{2}}$ is to $\overline{1-5n}|^{\frac{3}{2}}$. By proceeding in the ſame manner as above, it will eaſily appear, that n will come out greater than before, though s be ever ſo ſmall; and if y^4 be taken into the account, n will ſtill be greater.

From

From what has been ſaid, it is evident that the value of n, determined above, is not greater than it ought to be, and conſequently, that Dr *Stewart*'s value is too little, allowing all the principles he has gone upon to be right.

And by the bye, does it not appear probable, from the Doctor's being ſatisfied with a proportion for determining n, which, though very near the truth, is nevertheleſs inaccurate; that he has not been well apprized of the *great* alteration in the ſun's diſtance, which is occaſioned by a very ſmall one in the diſturbing force?—This is mentioned as a circumſtance which leads to the moſt favourable apology, that can be made, for the confidence with which our author gives us a meaſurement, of the ſolar ſyſtem, ſo much different from that which has been hitherto held, by our moſt eminent Aſtronomers,

nomers, as not egregiouſly wide of the truth.—To proceed:

But if our value of n, determined above, be ſubſtituted in the equation for determining a, deduced from what we call the true mean diſturbing force, we ſhall have a, or the diſtance of the ſun, ſtill much leſs than before; and indeed much leſs than the ſun's diſtance was ever thought to be; for it is only $= 50{\cdot}1$ times the diſtance of the moon, nearly.

Here then is a very large error, ariſing from a miſtake in the *Doctor's calculus*; it now remains that we examine the principles themſelves which he has aſſumed, and ſee whether we have any thing to expect from this method of finding the ſun's diſtance or not. Thoſe objections only will be mentioned which ſeem to be of the greateſt weight, nor ſhall we give any calculations, as it will be ſuffi-

cient

cient to point out where we apprehend he has erred.

In the firſt place, Dr *Stewart* ſuppoſes the body to deſcribe a revolution in the curve *ABC*, in the ſame time that another body would deſcribe the circle *AGD*, with the compound force at *A*; but that this is not ſo, has been proved, page 27.

2dly, Is it not eſſentially neceſſary in this enquiry, that the proportion of the ſolar force to that of the earth upon the moon, be exactly found? And in determining this, is not the periodic time of the moon round the earth taken into the account, and the earth ſuppoſed to be the center round which the moon moves? This therefore muſt occaſion a conſiderable error, the earth and moon, as it is well known, moving round their common center of gravity; and conſequently, a period of the moon will be accompliſhed in leſs time,

time, than if the earth were the center about which it revolved. To correct this properly, it will be neceſſary to know exactly the proportion of the quantities of matter in the earth and moon; a particular I am afraid we ſhall never be able perfectly to come at.

3dly, We have anticipated another capital objection, where we obſerved, that though the mean diſturbing force were ever ſo exactly found, yet the diſtance of the ſun could not be determined from it to any *great* degree of accuracy. [See page 23.]

4thly, Since the center of gravity of the earth and moon moves forward, in the ecliptic, ſeveral degrees every revolution of the moon round the earth, the motion of the apſes occaſioned by this diſturbing force, will be conſiderably different from what it would be, ſuppoſing the

the center of gravity at reſt, as we have done in this propoſition. This requires no demonſtration, being evident as ſoon as mentioned.

5thly, The true figure of the moon's orbit being an ellipſis, it muſt, with the ſame diſturbing force, have a different motion of the apſes to that of our ſuppoſed circular one; and to anſwer the queſtion right, the motion of the apſes ought to be determined in general for any excentricity whatever.—To ſee how the above conſiderations would ſucceed in calculation, I made a rude trial, but the reſult varied greatly both from Dr *Stewart*'s numbers, and thoſe commonly adopted by Aſtronomers.

6thly, But if the motion of the moon's apogee cannot be accounted for, upon the common principles of central forces, (as

is the opinion of Mr *Machine*) will not this be a more diſheartning circumſtance than any pointed out above?

From what has been ſhewn, it plainly appears, that the method of determining the ſun's diſtance, pitched upon by our author, is very ill fitted for the purpoſe; it being ſcarce poſſible, I think, to accompliſh it this way (that is, by the motion of the apſes) though the calculations were made with the greateſt accuracy. For the principles are too complex to make it poſſible to take every thing in to the account which belongs to it; and unleſs that be done, ſince the ſmalleſt neglect occaſions a very great error, the reſult will ever be much wide of the truth.

Neither is the motion of the nodes much better fitted than the motion of the

the apſes for the ſolution of this problem. In the firſt place, this motion is ſlower, and therefore requires a more accurate calculus. And beſides, the force $M\mathfrak{Q}$ (ſee fig. 1ſt) which moſtly contributes to the motion of the nodes, affects the diſtance of the ſun very much, by a ſmall variation. For if the value of it be put into fluxions, and reduced as was ſhewn above, we ſhall have the variation of this force to the variation of the ſun's diſtance, as $-4y^2+r^2$ to $\overline{a-2y}|^3$. Hence it is evident, that as the variation of the ſun's diſtance bears ſo very great a proportion to the variation of the diſturbing force; it is requiſite that this force be determined to the *laſt* degree of accuracy. But this will be very difficult, if not impoſſible to come at; the figure of the lunar orbit, its inclination to the ecliptic, &c. &c. being ſo variable.

From

From what has been ſaid, 'tis preſumed it may be ſafely concluded, that the diſtance of the ſun will never be ſatisfactorily aſcertained by the THEORY of GRAVITY.

FINIS.

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ERRATA.

Page 3 line 1, read $SM = \sqrt{a^2 - 2ay + r^2}$. P. 5 in the fig. at the interſection of the lines *BD*, *ML* place *F*. P. 13 l. 16, read *evidently*. P. 35 l. 9, read *affected*. Ib. line laſt, read $\frac{1}{3} \times \frac{4ny^3}{2ny - 1 - 5n \times y^3 \Big|^{\frac{1}{3}}}$.